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Analytical method for steady state vibration of system with localized non-linearities using convolution integral and Galerkin method

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Abstract

The analytical method using transfer function or impulse response is very effective for analyzing non-linear systems with localized non-linearities. This is because the number of non-linear equations can be reduced to that of the equations with respect to points connected with the non-linear element. In the present paper, analytical method for the steady state vibration of non-linear system including subharmonic vibration is proposed by utilizing convolution integral and the impulse response. The Galerkin method is introduced to solve the non-linear equations formulated by the convolution integral, and then the steady state vibration is obtained. An advantage of the present method is that stability or instability of the steady state vibration can be discriminated by the transient analysis from convolution integral. The three-degree-of-freedom mass–spring system is shown as a numerical example and the proposed method is verified by comparing with the result by Runge–Kutta–Gill method.
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1. Introduction

Non-linearities often exist at supported and connected points of mechanical structure, which are usually modelled by non-linear springs. However, their number is usually much smaller than that of the linear elements. For such systems, the analytical method using transfer function or impulse response is effective because the number of non-linear equations can be reduced to that of the equations with respect to points connected and supported with the non-linear elements, hence the scale of the non-linear analysis becomes small regardless of the number of degree-of-freedom

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of the system. In addition, when the transfer function or the impulse response is obtained for the existing system, the system damping can be easily modelled.

Vibration analyses have been reported using transfer function or the impulse response for such systems with localized non-linearities. Hagedorn et al. [1] showed that the transient vibration of such a system could be calculated by convolution integral with linear impulse response and non-linear restoring force. Chiang and Noah [2] proposed the nonlinear substructure synthetic method by use of convolution integral and transition matrix [3] and analyzed the transient vibration of rotor-housing system with bearing non-linear characteristics. Ren [4,5] derived non-linear equation from the transfer function synthesis method with the non-linear connecting force and obtained the steady state vibration and aperiodic vibration by harmonic balance method. Gordis and Radwick [6] reported non-linear substructure synthetic method using the non-linear Volterra integral equation. The non-linear equation for the steady state vibration using transfer function synthetic method can be commonly solved by the harmonic balance method; however, it is not always effective for predicting vibration response of the system in a practical manner, because of disability to discriminate stability and instability of the vibration, which is one of the important vibration property.

In the present paper, the analytical method using convolution integral is proposed for analyzing steady state vibration of the system with localized non-linearity including subharmonic vibration. The method also permits the stability–instability discrimination for the steady state vibration. The numerical results for fundamental and subharmonic vibration of three-degree-of-freedom system with localized non-linear spring are illustrated and the proposed method is verified in comparison with the result of Runge–Kutta–Gill (RKG) method.

2. Analytical method

2.1. System with localized non-linearities

The proposed method in the present study is very effective for the system with localized non-linearity. In the case of mass–spring system, for example, the non-linear system includes some masses that are supported at respective points by a non-linear spring as shown in Fig. 1, or masses that are connected to each other by non-linear spring. In this paper, the suggested method is applied to the former system for its simplicity. In Fig. 1, m_p represents a mass subjected to external force $f(t)$, m_q a mass supported with non-linear spring and m_r a mass of which vibration is being analyzed. Displacements of m_q and m_r are represented as x and y, respectively. $g(x)$ denotes

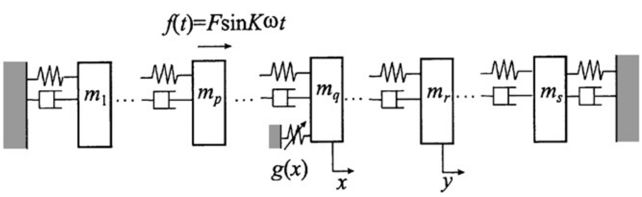

Fig. 1. Multi-degree-of-freedom system with localized non-linearity.

restoring force property of the non-linear spring. $f(t)$ is defined as $F \sin K\omega t$ so that the steady state vibration including subharmonic vibration of order $1/K$ can be considered.

2.2. Transient vibration

An outline of the method for transient vibration analysis using convolution integral is described in this section, which was reported by Hagedorn and Schramm [1] and provides the basis for the following steady state vibration analysis. When restoring force of the non-linear spring on m_q is regarded as external force, displacement x of m_q and displacement y of m_r can be determined by convolution integral as the following Eqs. (1) and (2), respectively,

$$x(t) = \int_0^t h^{qp}(t-\tau)f(\tau)\,\mathrm{d}\tau - \int_0^t h^{qq}(t-\tau)g\{x(\tau)\}\,\mathrm{d}\tau, \tag{1}$$

$$y(t) = \int_0^t h^{rp}(t-\tau)f(\tau)\,\mathrm{d}\tau - \int_0^t h^{rq}(t-\tau)g\{x(\tau)\}\,\mathrm{d}\tau, \tag{2}$$

where $h^{ab}(t)$ ($a = q$ or r, $b = p$ or q) is impulse response of the system without the non-linear spring $g(x)$ and superscripts a and b represent the location of response and excitation, respectively. The initial displacement and initial velocity of the system are set to zero. Eq. (1) related to m_q represents a non-linear equation with respect to x, on the other hand, Eq. (2) is calculated linearly for masses which are not supported by non-linear spring, where $x(t)$ is already known by the calculation of Eq. (1). When Eqs. (1) and (2) are expressed in discrete form, they become as follows:

$$x_n = \sum_{k=0}^{n-1} h^{qp}_{n-k} f_k \Delta t - \sum_{k=0}^{n-1} h^{qq}_{n-k} g(x_k)\Delta t, \tag{3}$$

$$y_n = \sum_{k=0}^{n-1} h^{rp}_{n-k} f_k \Delta t - \sum_{k=0}^{n-1} h^{rq}_{n-k} g(x_k)\Delta t. \tag{4}$$

Δt is a time interval of discretization process, where $t = n\Delta t$ and $\tau = k\Delta t$. The subscript in Eqs. (3) and (4) corresponds to the discretized time steps, where x_n denotes displacement at $t = n\Delta t$, for example. Eq. (3) is expressed as the following matrix form along with the initial displacement $x_0 = 0$.

$$\begin{Bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{Bmatrix} = \begin{bmatrix} h^{qp}_0 & 0 & \cdots & 0 \\ h^{qp}_1 & h^{qp}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h^{qp}_n & h^{qp}_{n-1} & \cdots & h^{qp}_0 \end{bmatrix} \begin{Bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{Bmatrix} \Delta t - \begin{bmatrix} h^{qq}_0 & 0 & \cdots & 0 \\ h^{qq}_1 & h^{qq}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h^{qq}_n & h^{qq}_{n-1} & \cdots & h^{qq}_0 \end{bmatrix} \begin{Bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_n) \end{Bmatrix} \Delta t. \tag{5}$$

x_n can be calculated from $x_0, x_1, \ldots$ and x_{n-1} in Eq. (5), since $h^{qp}_0 = h^{qq}_0 = 0$. Therefore, it is possible to obtain displacement x_n of a mass supported by non-linear spring via forward substitution procedure increasing n successively. The displacement y_n of a mass which is not supported by the non-linear spring can be easily calculated by substitution of $x_0, x_1, \ldots$ and x_{n-1} into Eq. (4).

2.3. Steady state vibration

The steady state vibration including subharmonic vibration of order $1/K$ has K times period of external force. When time history response due to the periodic external force is divided into every K period, Eq. (5) is rewritten as follows:

$$\begin{Bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{Bmatrix} = \begin{bmatrix} \mathbf{h}_0^{qp} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{h}_1^{qp} & \mathbf{h}_0^{qp} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{h}_n^{qp} & \mathbf{h}_{n-1}^{qp} & \cdots & \mathbf{h}_0^{qp} \end{bmatrix} \begin{Bmatrix} \mathbf{f} \\ \mathbf{f} \\ \vdots \\ \mathbf{f} \end{Bmatrix} \Delta t - \begin{bmatrix} \mathbf{h}_0^{qq} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{h}_1^{qq} & \mathbf{h}_0^{qq} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{h}_n^{qq} & \mathbf{h}_{n-1}^{qq} & \cdots & \mathbf{h}_0^{qq} \end{bmatrix} \begin{Bmatrix} \mathbf{g}(\mathbf{x}_0) \\ \mathbf{g}(\mathbf{x}_1) \\ \vdots \\ \mathbf{g}(\mathbf{x}_n) \end{Bmatrix} \Delta t, \qquad (6)$$

where the subscript shows the order of every K period. When a time interval corresponding to each K period of the external force is divided into M equal sections, $\mathbf{x}_i$ and $\mathbf{f}$ are expressed as column vectors which consist of M elements and $\mathbf{g}(\mathbf{x}_i)$ a column vector which consists of M elements of restoring force $g(x_i)$

$$\mathbf{x}_i = \begin{Bmatrix} x_{i\times M} \\ x_{i\times M+1} \\ \vdots \\ x_{(i+1)M-1} \end{Bmatrix}, \quad \mathbf{f} = F \begin{Bmatrix} \sin 2\pi K \frac{0}{M} \\ \sin 2\pi K \frac{1}{M} \\ \vdots \\ \sin 2\pi K \frac{M-1}{M} \end{Bmatrix}, \quad \mathbf{g}(\mathbf{x}_i) = \begin{Bmatrix} g(x_{i\times M}) \\ g(x_{i\times M+1}) \\ \vdots \\ g(x_{(i+1)M-1}) \end{Bmatrix}. \qquad (7)$$

$\mathbf{h}_i^{ab}$ is $M \times M$ matrix which consists of the impulse response h_j^{ab} , written as

$$\mathbf{h}_0^{ab} = \begin{bmatrix} h_0^{ab} & 0 & \cdots & 0 \\ h_1^{ab} & h_0^{ab} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ h_{M-1}^{ab} & h_{M-2}^{ab} & \cdots & h_0^{ab} \end{bmatrix}, \quad \mathbf{h}_i^{ab} = \begin{bmatrix} h_{i\times M}^{ab} & \cdots & \cdots & h_{(i-1)M+1}^{ab} \\ h_{i\times M+1}^{ab} & h_{i\times M}^{ab} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ h_{(i+1)M-1}^{ab} & h_{(i+1)M-2}^{ab} & \cdots & h_{i\times M}^{ab} \end{bmatrix} \quad (i \geqslant 1). \qquad (8)$$

$\mathbf{x}_i$ ($i = 0, 1, \ldots, n$) in Eq. (6) are individually different vectors because $\{\mathbf{x}_0^{\mathrm{T}}, \mathbf{x}_1^{\mathrm{T}}, \ldots, \mathbf{x}_n^{\mathrm{T}}\}$ corresponds to the transient vibration derived from Eq. (5), where ($^{\mathrm{T}}$) denotes transposed matrix. When the time has sufficiently passed, it is considered that the transient vibration converges to steady state vibration. Then Eq. (6) can be rewritten under such condition as follows:

$$\begin{Bmatrix} \vdots \\ \vdots \\ \mathbf{x} \\ \vdots \\ \mathbf{x} \\ \mathbf{x} \end{Bmatrix} = \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \ddots & \ddots & \mathbf{0} & \ddots & \ddots & \vdots \\ \ddots & \ddots & \mathbf{h}_0^{qp} & \mathbf{0} & \ddots & \vdots \\ \ddots & \ddots & \ddots & \mathbf{h}_0^{qp} & \mathbf{0} & \vdots \\ \ddots & \mathbf{h}_i^{qp} & \ddots & \ddots & \mathbf{h}_0^{qp} & \mathbf{0} \\ \ddots & \mathbf{h}_{i+1}^{qp} & \mathbf{h}_i^{qp} & \ddots & \ddots & \mathbf{h}_0^{qp} \end{bmatrix} \begin{Bmatrix} \vdots \\ \vdots \\ \mathbf{f} \\ \vdots \\ \mathbf{f} \\ \mathbf{f} \end{Bmatrix} \Delta t - \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \mathbf{0} \\ \ddots & \ddots & \mathbf{0} & \ddots & \ddots & \vdots \\ \ddots & \ddots & \mathbf{h}_0^{qq} & \mathbf{0} & \ddots & \vdots \\ \ddots & \ddots & \ddots & \mathbf{h}_0^{qq} & \mathbf{0} & \vdots \\ \ddots & \mathbf{h}_i^{qq} & \ddots & \ddots & \mathbf{h}_0^{qq} & \mathbf{0} \\ \ddots & \mathbf{h}_{i+1}^{qq} & \mathbf{h}_i^{qq} & \ddots & \ddots & \mathbf{h}_0^{qq} \end{bmatrix} \begin{Bmatrix} \vdots \\ \vdots \\ \mathbf{g}(\mathbf{x}) \\ \vdots \\ \mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}) \end{Bmatrix} \Delta t, \tag{9}$$

where $\mathbf{x}$ is a vector corresponding to time history response of the steady state vibration. If the system has damping, the impulse response $h^{ab}(t)$ converges to zero after adequate course of time. Hence $\mathbf{h}_i^{ab}$ which consists of the impulse response data is regarded as zero when the subscript is larger than N, that is, $\mathbf{h}_N^{ab} = \mathbf{h}_{N+1}^{ab} = \cdots = \mathbf{0}$. Therefore, the next Eq. (10) can be obtained by calculating bottom row of Eq. (9)

$$\mathbf{x} = \mathbf{H}^{qp}\mathbf{f} - \mathbf{H}^{qq}\mathbf{g}(\mathbf{x}), \tag{10}$$

where $\mathbf{H}^{ab}$ is represented as

$$\mathbf{H}^{ab} = \sum_{i=0}^{N-1} \mathbf{h}_i^{ab} \Delta t. \tag{11}$$

N is the smallest value of i which satisfies the following relationship:

$$\frac{\sqrt{\sum_{k=i\times M}^{(i+1)M-1} (h_k^{ab})^2}}{\sqrt{\sum_{k=0}^{i\times M-1} (h_k^{ab})^2}} \leqslant \varepsilon. \tag{12}$$

Eq. (12) signifies that the ratio between the magnitude (square root of the squared sum) at a single $(i+1)$th period of the impulse response and the magnitude up to the ith period becomes smaller than ε. Note that N for h_i^{qp} are different from N for h_i^{qq}. $\varepsilon = 10^{-5}$ is adopted in the following example. The equation in $\mathbf{y}$, which determines the steady state vibration of m_r, is obtained through the same procedure expressed as follows:

$$\mathbf{y} = \mathbf{H}^{rp}\mathbf{f} - \mathbf{H}^{rq}\mathbf{g}(\mathbf{x}). \tag{13}$$

If Eq. (10) is solved for $\mathbf{x}$, the time history of steady state vibration x is obtained, so that y can be calculated by substituting $\mathbf{x}$ into Eq. (13). However, generally it is difficult to solve such simultaneous non-linear equations containing M unknowns as Eq. (10). Therefore, the Galerkin method is adopted in order to acquire a approximate solution, which is explained below.

When the subharmonic vibration of order $1/K$ is assumed, the solution of Eq. (10) is expressed as follows:

$$\mathbf{x} = A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K, \tag{14}$$

where $\mathbf{S}_1$, $\mathbf{C}_1$, $\mathbf{S}_K$ and $\mathbf{C}_K$ represent the following vectors, which consist of sine or cosine function divided into M terms equally

$$\mathbf{S}_1 = \begin{Bmatrix} \sin 2\pi\frac{0}{M} \\ \sin 2\pi\frac{1}{M} \\ \vdots \\ \sin 2\pi\frac{M-1}{M} \end{Bmatrix}, \quad \mathbf{C}_1 = \begin{Bmatrix} \cos 2\pi\frac{0}{M} \\ \cos 2\pi\frac{1}{M} \\ \vdots \\ \cos 2\pi\frac{M-1}{M} \end{Bmatrix},$$

$$\mathbf{S}_K = \begin{Bmatrix} \sin 2\pi K\frac{0}{M} \\ \sin 2\pi K\frac{1}{M} \\ \vdots \\ \sin 2\pi K\frac{M-1}{M} \end{Bmatrix}, \quad \mathbf{C}_K = \begin{Bmatrix} \cos 2\pi K\frac{0}{M} \\ \cos 2\pi K\frac{1}{M} \\ \vdots \\ \cos 2\pi K\frac{M-1}{M} \end{Bmatrix}. \tag{15}$$

A_1 and B_1 denote magnitude of the subharmonic component of order $1/K$, and A_K and B_K magnitude of the fundamental harmonic component. Hereby, the second equation of Eqs. (7) yields the expression of $\mathbf{f} = F\mathbf{S}_K$. Substituting Eq. (14) into Eq. (10) and multiplying $\mathbf{S}_1^{\mathrm{T}}$, $\mathbf{C}_1^{\mathrm{T}}$, $\mathbf{S}_K^{\mathrm{T}}$ and $\mathbf{C}_K^{\mathrm{T}}$, respectively, on each term from its left-hand side, the following equations are obtained:

$$\begin{aligned}
\frac{M}{2}A_1 &= F\mathbf{S}_1^{\mathrm{T}}\mathbf{H}^{qp}\mathbf{S}_K - \mathbf{S}_1^{\mathrm{T}}\mathbf{H}^{qq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K), \\
\frac{M}{2}B_1 &= F\mathbf{C}_1^{\mathrm{T}}\mathbf{H}^{qp}\mathbf{S}_K - \mathbf{C}_1^{\mathrm{T}}\mathbf{H}^{qq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K), \\
\frac{M}{2}A_K &= F\mathbf{S}_K^{\mathrm{T}}\mathbf{H}^{qp}\mathbf{S}_K - \mathbf{S}_K^{\mathrm{T}}\mathbf{H}^{qq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K), \\
\frac{M}{2}B_K &= F\mathbf{C}_K^{\mathrm{T}}\mathbf{H}^{qp}\mathbf{S}_K - \mathbf{C}_K^{\mathrm{T}}\mathbf{H}^{qq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K).
\end{aligned} \tag{16}$$

A_1, B_1, A_K and B_K can be calculated from Eqs. (16) by the Newton–Raphson method, and the time history of x can be obtained from Eq. (14). If A_K and B_K are calculated by Eqs. (16) under the condition of $A_1 = B_1 = 0$, fundamental harmonic solution is obtained.

The solution $\mathbf{y}$ of Eq. (13) is also expressed as follows:

$$\mathbf{y} = C_1\mathbf{S}_1 + D_1\mathbf{C}_1 + C_K\mathbf{S}_K + D_K\mathbf{C}_K. \tag{17}$$

Substituting Eq. (17) into Eq. (13) and multiplying $\mathbf{S}_1^{\mathrm{T}}$, $\mathbf{C}_1^{\mathrm{T}}$, $\mathbf{S}_K^{\mathrm{T}}$ and $\mathbf{C}_K^{\mathrm{T}}$, respectively, on each term of equation, the following equations are obtained:

$$\begin{aligned}
\frac{M}{2}C_1 &= F\mathbf{S}_1^{\mathrm{T}}\mathbf{H}^{rp}\mathbf{S}_K - \mathbf{S}_1^{\mathrm{T}}\mathbf{H}^{rq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K),\\
\frac{M}{2}D_1 &= F\mathbf{C}_1^{\mathrm{T}}\mathbf{H}^{rp}\mathbf{S}_K - \mathbf{C}_1^{\mathrm{T}}\mathbf{H}^{rq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K),\\
\frac{M}{2}C_K &= F\mathbf{S}_K^{\mathrm{T}}\mathbf{H}^{rp}\mathbf{S}_K - \mathbf{S}_K^{\mathrm{T}}\mathbf{H}^{rq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K),\\
\frac{M}{2}D_K &= F\mathbf{C}_K^{\mathrm{T}}\mathbf{H}^{rp}\mathbf{S}_K - \mathbf{C}_K^{\mathrm{T}}\mathbf{H}^{rq}\mathbf{g}(A_1\mathbf{S}_1 + B_1\mathbf{C}_1 + A_K\mathbf{S}_K + B_K\mathbf{C}_K).
\end{aligned} \tag{18}$$

A_1, B_1, A_K and B_K obtained from Eqs. (16) are substituted into Eq. (18), hence C_1, D_1, C_K and D_K, that is, the steady state vibration of y, can be determined.

3. Discrimination of stability and instability

The steady state vibration including subharmonic vibration is determined from Eq. (10) as shown in the previous section but both stable and unstable steady state vibration are obtained since the periodic solution is assumed as shown in Eq. (14). Discrimination method for stability and instability of the steady state vibration is described in this section, where the convolution integral is used to calculate the transient vibration.

The discrimination can be performed by investigating the transient response from the initial condition equal to the steady state vibration. If it diverges from the initial steady state vibration and then converges to other steady state, the initial steady state vibration is regarded as unstable, otherwise it is discriminated as stable steady state. The transient vibration for a given initial steady state solution can be calculated by the procedure for transient vibration as already described in the Section 2.2. Such calculation is expressed as follows:

$$\begin{aligned}
\begin{Bmatrix} \mathbf{x} \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_i \end{Bmatrix} &= \begin{bmatrix} \mathbf{h}_{N'-1}^{qp} & \cdots & \mathbf{h}_0^{qp} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_{N'-1}^{qp} & \cdots & \mathbf{h}_0^{qp} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{h}_{N'-1}^{qp} & \cdots & \mathbf{h}_0^{qp} \end{bmatrix} \begin{Bmatrix} \mathbf{f} \\ \vdots \\ \mathbf{f} \\ \mathbf{f} \\ \vdots \\ \mathbf{f} \end{Bmatrix} \Delta t \\
&\quad - \begin{bmatrix} \mathbf{h}_{N-1}^{qq} & \cdots & \mathbf{h}_0^{qq} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_{N-1}^{qq} & \cdots & \mathbf{h}_0^{qq} & \cdots & \mathbf{0} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{h}_{N-1}^{qq} & \cdots & \mathbf{h}_0^{qq} \end{bmatrix} \begin{Bmatrix} \mathbf{g}(\mathbf{x}) \\ \vdots \\ \mathbf{g}(\mathbf{x}) \\ \mathbf{g}(\mathbf{x}_1) \\ \vdots \\ \mathbf{g}(\mathbf{x}_i) \end{Bmatrix} \Delta t,
\end{aligned} \tag{19}$$

where the equations of the transient vibration follow the bottom row in Eq. (9), and the convergence of impulse response to become zero is considered, that is, $\mathbf{h}_n^{qp} = 0$ $(n \geqslant N')$ and $\mathbf{h}_n^{qq} = 0$ $(n \geqslant N)$.

Since $\mathbf{x}$ in Eq. (19) is already obtained as steady state solution, the calculation of the transient vibration starts from $\mathbf{x}_1$ of the second row in Eq. (19) and successively the time history response $\mathbf{x}_i$ can be calculated by the forward substitution procedure since the diagonal elements of $\mathbf{h}_0^{\mathrm{qq}}$ are zero. When A_1', B_1', A_K' and B_K' represent magnitude coefficients of $\sin\omega t$, $\cos\omega t$, $\sin K\omega t$ and $\cos K\omega t$ components of $\mathbf{x}_i$, respectively, they are obtained from the following equations:

$$A_1' = \frac{2}{M}\mathbf{S}_1^{\mathrm{T}}\mathbf{x}_i, \quad B_1' = \frac{2}{M}\mathbf{C}_1^{\mathrm{T}}\mathbf{x}_i, \quad A_K' = \frac{2}{M}\mathbf{S}_K^{\mathrm{T}}\mathbf{x}_i, \quad B_K' = \frac{2}{M}\mathbf{C}_K^{\mathrm{T}}\mathbf{x}_i. \tag{20}$$

In the case of subharmonic vibration of order $1/K$, A_1' and B_1' are compared with A_1 and B_1 of the steady state vibration, hereby it is regarded as stable if the following relationship is satisfied:

$$\frac{\sqrt{(A_1 - A_1')^2 + (B_1 - B_1')^2}}{\sqrt{A_1^2 + B_1^2}} \leqslant \frac{1}{10}. \tag{21}$$

Eq. (21) denotes that both the difference between A_1 and A_1' and also the difference between B_1 and B_1' are small. The stability condition in right side of Eq. (21) seems comparatively moderate, since the difference between the approximate solution of A_1 and B_1 obtained by the Galerkin method and the exact solution $\mathbf{x}_i$ by the transient calculation frequently becomes large. In the case of the fundamental harmonic vibration, the following condition with A_K, A_K', B_K and B_K' is adopted

$$\frac{\sqrt{(A_K - A_K')^2 + (B_K - B_K')^2}}{\sqrt{A_K^2 + B_K^2}} \leqslant \frac{1}{10}. \tag{22}$$

4. Example for three degree-of-freedom mass–spring system

A numerical example on fundamental harmonic vibration and subharmonic vibration of order $\frac{1}{3}$ in three-degree-of-freedom mass–spring system of Fig. 2 is shown. m_1 is subjected to sinusoidal wave force and m_2 is supported by a non-linear spring. The property of non-linear spring is

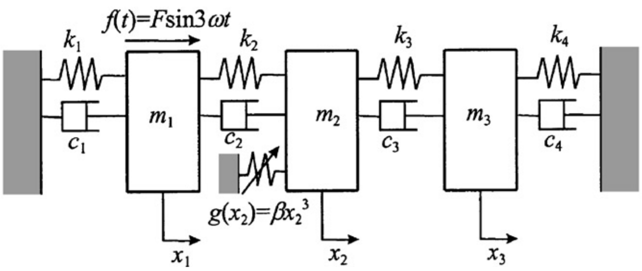

Fig. 2. Three-degree-of-freedom mass–spring system.

Table 1
System parameters

Mass (kg)	Damping coefficient (N s/m)	Spring constant (N/m)	Non-linear spring constant (N/m^3)	Amplitude of excitation (N)
$m_1 = 1.0$	$c_1 = 1.0$	$k_1 = 1000$	$\beta = 1.0 \times 10^8$	$F = 1.0$
$m_2 = 2.0$	$c_2 = 1.0$	$k_2 = 1000$		
$m_3 = 3.0$	$c_3 = 1.0$	$k_3 = 1000$		
	$c_4 = 1.0$	$k_4 = 1000$		

defined as $g(x) = \beta x^3$. The system parameters used in the example are shown in Table 1. The case $K = 3$ is illustrated so that the subharmonic vibration of order $\frac{1}{3}$ is considered. The number of division in three periods is set to $M = 60$. The impulse response of each mass is determined by modal analysis on free vibration for the corresponding given initial velocity.

Response curves of the fundamental harmonic vibration and the subharmonic vibration of order $\frac{1}{3}$ on m_1, m_2 and m_3 are shown in Fig. 3(a)–(c), respectively. The dotted line denotes amplitude $\sqrt{A_3^2 + B_3^2}$ and phase $\tan^{-1}(B_3/A_3)$ of the fundamental harmonic vibration, and also the solid line denotes amplitude $\sqrt{A_1^2 + B_1^2}$ and phase $\tan^{-1}(B_1/A_1)$ of the subharmonic vibration. The amplitude and the phase of the fundamental component in the subharmonic vibration almost coincide with those of the fundamental harmonic vibration, hence they are omitted in the figures. There are three solutions for the subharmonic vibration which have the same amplitude, whereas the phase has difference of $2\pi/3$ rad, respectively. In Fig. 4, the calculation result of m_2 using the convolution integral is compared with the result of the RKG method for equations of motion of the system shown in Fig. 2, where the amplitude is represented as RMS value. Furthermore, stability and instability of the steady state vibration are discriminated in Fig. 4. It is found that both results sufficiently coincide on the stable vibration.

Calculation result of stability analysis for the subharmonic vibration is represented schematically in Fig. 5. Co-ordinates of A_1' and B_1' are the coefficients of $\sin \omega t$ and $\cos \omega t$ terms calculated in Eq. (20), and their values on transient vibration are plotted as dotted line in Fig. 5 along with the increase of time. If the steady state vibration is stable, location (A_1', B_1') does not move, while it leaves the steady state point if unstable. However, even if stable, the convergent point of the transient vibration differs from the stable steady state vibration point obtained by the convolution integral for the reason as already described in Section 3. The calculation result for the case 8.29 Hz is shown in Fig. 5(a), where the unstable subharmonic vibration of order $\frac{1}{3}$ converges to another stable one. In the case of 9.00 Hz which is shown in Fig 5(b), the unstable vibration converges to the fundamental vibration.

In order to discuss the accuracy of the Galerkin method, Eq. (10) is solved directly by the Newton–Raphson method (which is called the direct method henceforth) and the result is compared with that of the Galerkin method. The time history response obtained by the Galerkin method is given as initial value for $\mathbf{x}$ in the direct method. Comparison between both waves of $\frac{1}{3}$ order subharmonic vibrations is shown in Fig. 6. For the case 8.70 Hz which is shown in Fig. 6(a), the Galerkin method (dotted line) is slightly different from the direct method (solid line). However, both the lines almost coincide in Fig. 6(b) of 9.24 Hz case. Both waves coincide on most

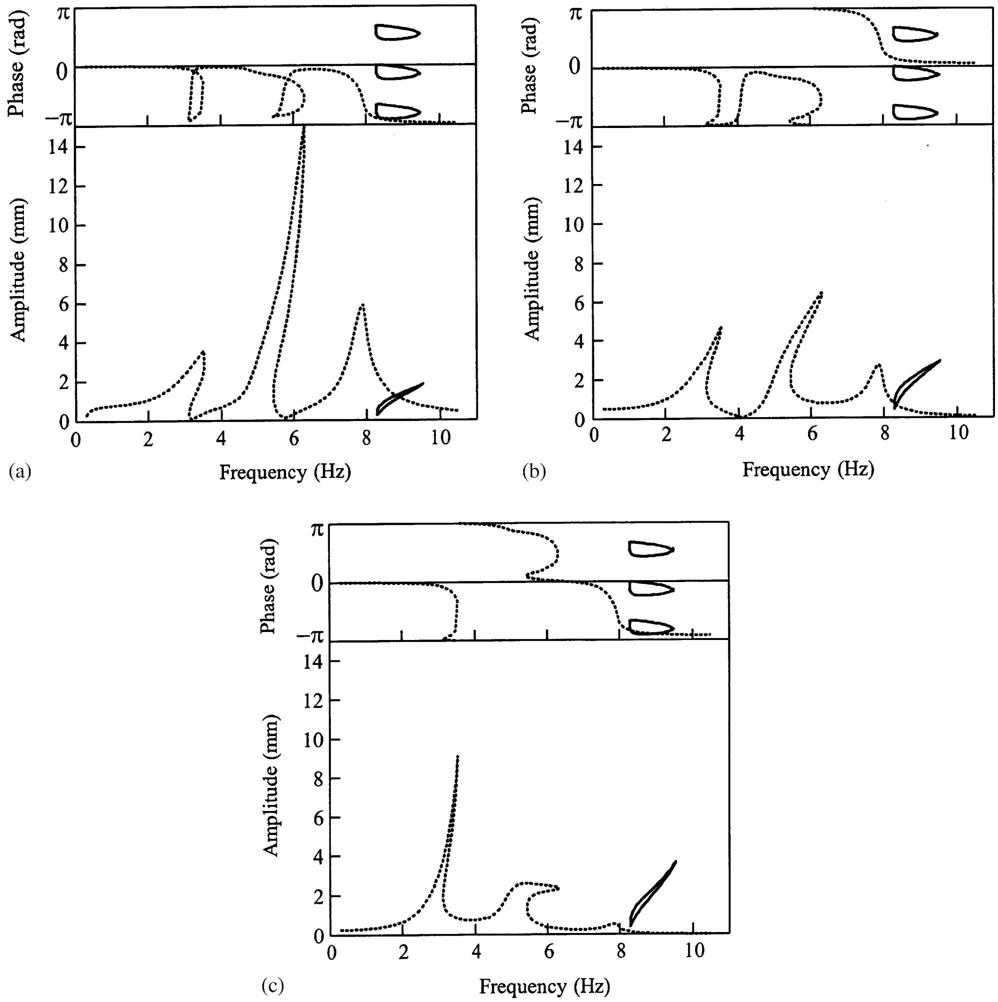

Fig. 3. Response curves. (a) m_1, (b) m_2, and (c) m_3. (…) Fundamental harmonic vibration, (—) $\frac{1}{3}$ subharmonic vibration component.

region of the response curve, while the case where both waves do not coincide is limited in the vicinity of an edge of the subharmonic region. It is considered that the overall system with localized non-linearity exhibits a weak non-linearity, although the localized non-linearity is strong. Therefore, valid solution of Eq. (10) can be determined by the Galerkin method with assumption of the simple solution such as Eq. (14).

5. Conclusions

In this paper, the analytical method of steady state vibration using convolution integral is proposed for the system with localized non-linearities and it is shown that the non-linear equation

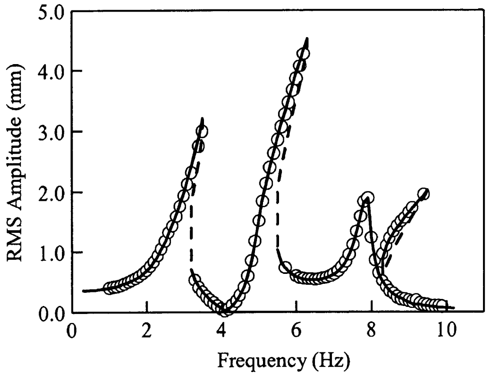

Fig. 4. Comparison with RKG method. (—) Stable vibration by convolution integral, (-----) unstable vibration by convolution integral, (○) stable vibration by RKG method.

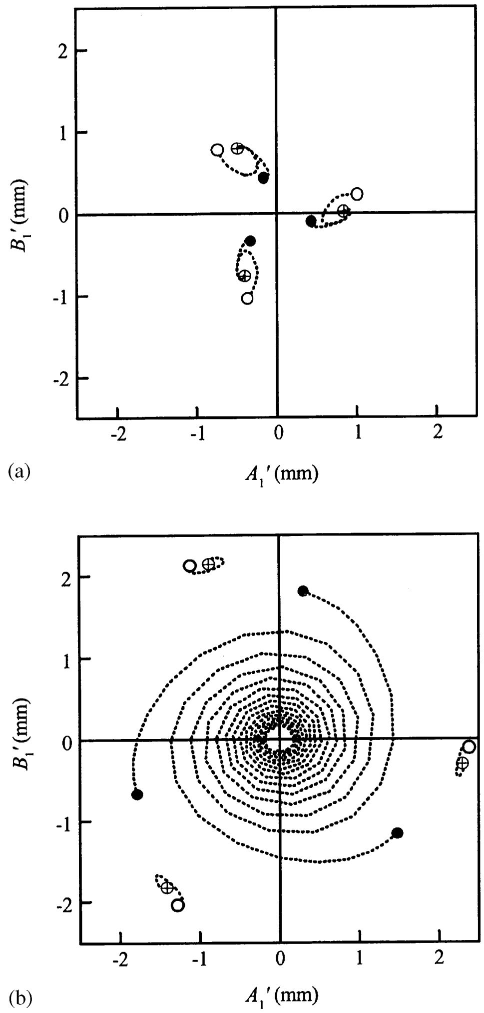

Fig. 5. Stability and instability in subharmonic vibration of order $\frac{1}{3}$. (a) 8.29 Hz and (b) 9.00 Hz. (…) Orbit of transient vibration, (○) stable vibration, (•) unstable vibration, (⊕) convergent point.

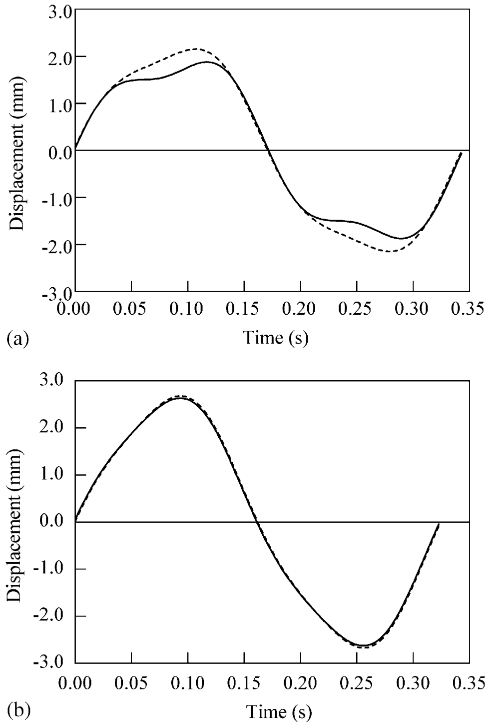

Fig. 6. Comparison of direct method and Galerkin approximation. (a) 8.70 Hz and (b) 9.24 Hz. (—) Direct method, (…) Galerkin method.

formulated by the suggested analytical process can be solved by the Galerkin method. Subharmonic vibration is also included in the steady state vibration. Furthermore, the stability of each solution can be discriminated by the transient analysis using the convolution integral. Numerical example of three-degree-of-freedom mass–spring system is demonstrated and the following results are obtained. (1) Response curves of the fundamental harmonic vibration and the subharmonic vibration of order $\frac{1}{3}$ can be easily calculated by the present method. (2) Stability or instability of the steady state vibration can be discriminated by the transient analysis using convolution integral. (3) The result obtained from the convolution integral coincides well with the result by the Runge–Kutta–Gill method. Therefore, the present method is efficient for the steady state vibration analysis of non-linear system with localized non-linearities.

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